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XIII.

On the Measure of the Forces of Bodies moving with different Velocities.

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[This paper was read to the Academy in March, 1854, as here printed, except a few sentences since added by way of illustration (as that from Brewster's "Life of Newton"), and some trifling verbal alterations.]

Although the great controversy on the measure of the forces of moving bodies, which divided the investigators of mechanical science during a part of the seventeenth and eighteenth centuries, is no longer considered open, as a subject for scientific speculation or inquiry, by many of the leading writers of the present day, they having concluded that it has been exhausted, or rather that it had terminated in the discovery that the whole question was of a verbal rather than of a substantial character, I have yet ventured to present to the Academy a paper touching it. I am not aware that any one has heretofore followed the course of experiment, and the particular reasoning from it, which will be found in this paper, though the force of the spring has often been appealed to by partisans on both sides of the controversy, as proving the truth of the standard which each attempted to establish. I shall be very willing, however, to forego all claim to originality, if what is herein presented shall be found to assist any one to a clearer understanding of a subject on which Newton and Leibnitz were divided. Notwithstanding the high authority which, in more recent times, has decided that their disagreement was rather one of words than of things, I cannot but consider it as substantial and irreconcilable, not only affecting all just conceptions of mechanical force, but leading even those who thus attempt to reconcile the opposing theories into very grave practical errors and contradictions.

Thus Dr. Lardner says ("Mechanics," chap. iv.): "If a cannon-ball were forty times the weight of a musket-ball, but the musket-ball moved with forty times the velocity of the cannon-ball, both would strike any obstacle with the same force, and would overcome the same resistance"; — a statement that denies to the musket-ball *thirty-nine fortieths* of its force. I do not cite Dr. Lardner as an authority of a very high order, but as a leading lecturer and a writer of standard systematic works. But no one will, I presume, object to Prof. John Playfair as a representative of the modern writers who hold to the merely verbal character of this controversy. He was not only, according to Lord Cockburn (in the "Edinburgh Review"), "the best philosophical writer in the English language," but he made this controversy a subject of special study; and he came to the conclusion, that, if the proper precautions were taken in limiting the vagueness and ambiguity of the *data*, there would be no difference in the practical applications of force, and "that the propositions maintained by both sides were true, and were not opposed to each other." But let us see how he succeeded in applying these not opposed theories. In his "Outlines of Natural Philosophy," his last systematic work, published many years after the assertions cited above, he says (Vol. I. p. 206): "If the sections of two streams be the same, the forces with which they will strike on planes directly opposed to them are as the squares of their velocities. For the force of a stream must be as the force of each particle and as the number of particles that strike in a given time. Now, the force of each particle is as the velocity of the fluid, and the number of particles that strike in a given time, the section being given, is also as that velocity. Therefore the whole force of the stream must be as the square of its velocity." Not only is the assertion, that "the force of each particle is as the velocity of the fluid," directly opposed to the theory of Leibnitz, but the theorem, that "the whole force of the stream must be as the square of the velocity," is equally contradicted by the experiments of Smeaton. For, according to these writers, the force of each particle must be as the square, and that of the stream as the cube, of the velocity. Again, Playfair says (on page 216 of the same volume): "If the sections of two streams are the same, their forces will be as the squares of their velocities, or as the heights due to the velocity of the water, and the effects of the machines driven by them will be as the cubes of their velocities, or as the heights due to those velocities multiplied into their square roots." That is, *forces* which are known to us only as *causes* produce *effects* as much greater than themselves as the cube of a number is greater than its square.

With such instances of substantial and radical error, from substituting the conclusions of one theory for those of the other, how can it be said that the difference is merely verbal, or "that the propositions maintained by both sides were true, and were not opposed to each other"?

Again, in the late "Life of Newton" by Sir David Brewster, it is stated, that "Poleni had published an experiment which he considered as proving Leibnitz's assertion that the force of moving bodies is proportional to the square of the velocity, and not, *as is commonly thought*, to the simple velocity. Dr. Pemberton saw its insufficiency, and drew up a *refutation* of it." This being "communicated to Sir Isaac, he was so well pleased with it, that he called upon Pemberton at his lodgings, and showed him a refutation of Poleni, by himself, grounded upon other principles." This *refutation*, as it is called, is published in the "Philosophical Transactions," and is intended to support in its fullest extent the Newtonian measure of force, viz. that of the simple velocity, against the measure proposed by Leibnitz. Here then we have Sir David Brewster, one of the greatest of living names in mechanical science, in one of his latest works, asserting that the force of a moving body *is commonly thought* to be as its velocity *simply*, and that the assertion of Leibnitz that the force is proportional to the square of the velocity was *refuted* by Dr. Pemberton, and again by Newton himself.

While a question of this vital character, affecting all the conceptions and many of the computations of force, is in this contradictory and cloudy state, ought scientific men to rest upon it, or to suffer it to rest, as a closed subject? The labor of preparing the following paper shows my own opinion upon this question.

It may be well, perhaps, before opening the particular course of investigation pursued in this paper, to state briefly the precise matter that formed the subject of this renowned controversy. Suppose, then, a body, as a stone, A, of a certain mass, to be thrown in any direction in space, with a certain velocity, it must move onward with a certain force.* Now if another stone, B, having twice the mass of A, be thrown with the same velocity that was given to A, it must move with twice the force of A; for we may divide B so as to form two stones, each of which shall have a mass equal to that of A, and if thrown with the velocity of A, they must each have the force of A, and if each separately has the force of A, both, united to form B, must have twice the force of A. This is substantially the mode of reasoning used by Galileo to show that the mass of a body could not alter the velocity with which it would descend by its weight, when unsupported, to the earth; and it follows from this, that under a given velocity the forces of all bodies must be directly as their masses. If, however, instead of throwing different masses with one and the same velocity, we throw one and the same mass with different velocities, we search in vain for any reasoning of this simple and decisive kind to prove that the forces must be as the velocities, or to show the relation that one must bear to the other. For while we can thus unite two distinct masses, making them to coexist under a common velocity, we cannot unite two distinct velocities, making them to coexist in a common mass. With the hope, then, of aiding to form a clearer conception of this subject, let me propose the following experiment.

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* *Force* is the unknown cause by which a body resists change from, or produces change in, another body. *Synonyme*, Power. I shall avoid, if possible, the use of the words *momentum*, *vis mortua*, and *vis viva*, as they are often used and understood in such vague and indefinite senses as to obscure rather than to give any clearness to our conceptions in an inquiry like this.

Take a spiral spring, formed of eleven turns of wire, coiled so as to leave an open space between each turn of the coil (as in Fig. 2). Let this coil be then pressed together as in Fig. 1, and place against its ends respectively the weights A and B, suspended by cords, like pendulums. Let the weight of A be 10 pounds and the weight of B be 1 pound. On releasing the spring from its constrained condition it recovers, under the operation of its elasticity, its original form as shown in Fig. 2, while the masses A and B are thrown, one to the right and one to the left, with velocities inversely as their masses, namely, A with a velocity of 1, and B with a velocity of 10. All the force which had been applied to compress the spring, from whatever source it was derived, and which existed in the spring's elasticity, has now passed away from the spring, and been exhausted in giving motion to A and B. It has overcome, in those bodies, to a certain extent, their state of rest, and the force, *vis inertiæ*, with which they resisted all change of that state; and those bodies together now possess a force of motion, or inertia of motion, equal to the force of elasticity given out from the spring. We may therefore conceive that the force of elasticity in the spring has been transferred to A and B in the form of

Fig. 1.

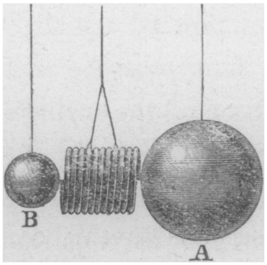
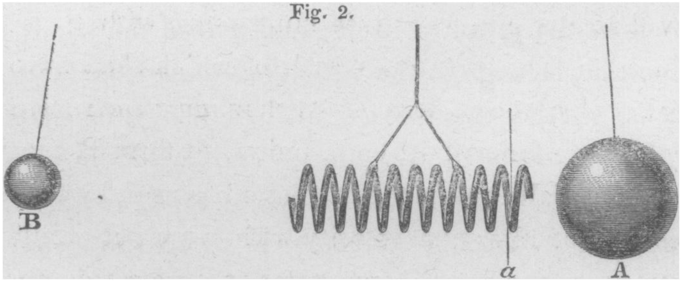


Fig. 2.



the force of the inertia of motion. This conception, while it violates no fact or result of experiment, will aid us in explaining them all.

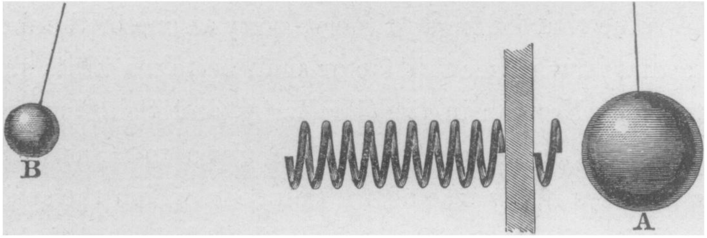
The question now is, How much of this force has gone to A, and how much to B? Or, What is the force of A and of B respectively, at the instant they are discharged from the spring? One party in the controversy, who array themselves under the great name of Newton, determine the force of each by multiplying its mass by its velocity, simply, and find A, mass 10, multiplied by its velocity 1, to be 10; and B, mass 1, multiplied by its velocity 10, to be 10. That is, the force of the spring is equally distributed between the two bodies. They hold that this must be so of necessity; for the spring during its action pressed equally to the right and to the left, equally against the bodies A and B. This pressure was continued an equal time upon both bodies, namely, during all the time of the spring's action. How is it possible then, say they, that a greater effect has been produced in one direction than in the other?

This affirmation that the action is equal upon both bodies may be held as not merely derived from, but as constituting the substance of, Newton's Third Law, which asserts that "to every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts"; and it is not to be denied that, so far as pressure constitutes action, this has been fully established by his citation of instances under this Third Law. But it by no means follows, because the spring maintains a like pressure by both its ends during the act of extending itself, that therefore its force is transferred equally to both bodies. The velocity with which this pressure overcomes the resistance opposed to it must be regarded, as well as the pressure itself; and indeed the whole controversy lies in determining the value that belongs to the velocity as one of the factors by which the force is produced; that is, whether this factor shall be taken in its simple form, a value clearly assigned to it by Newton in the Corollaries and Scholium of his Third Law; or whether a higher value, as its square, shall be given to it, as was first proposed by Leibnitz. Let us then see if, by tracing the course in which the spring elongates itself under different conditions, we can increase our knowledge of the transference of its force. By an examination of this subject, we shall find *that the spring acts upon each body with a velocity proportional to the resistance that it receives from the opposite body;* and we shall find *that, as we increase the mass of either body, the effect produced upon the opposite body will be increased.* Thus, if we add to the mass of B, until it becomes equal to that of A, and let the spring expand as before, the velocities of A and B will be equal; the velocity of A being more than double (as $\sqrt{5.5}$ to $\sqrt{1}$) what it was before the size of B was increased. The force expended, namely, the elastic force of the spring, has been the same in both cases. Will it be said that it has been distributed in the same proportion to the two bodies, when A, the mass of which has not been changed, has had its velocity increased in this essential degree ? And yet the Newtonians would make it the same; namely, half to each body in both instances.

We may, I think, obtain a clear conception of the action of the spring, if we consider it as depending, for its effect on each body, upon the degree of immobility which it finds in the opposite body; and hence *the force communicated to each body will depend upon the resistance of the inertia of the opposite body ;* or, *through the action of the spring, the inertia of rest in each body is transferred as an active force in the motion of the opposite body.* In the case before us, the body A having an inertia 10 times as great as that of B, the force communicated to B by the spring is 10 times as great as the force communicated to A. To show this more clearly, let us attend to the exact course of the action of the spring. It will be found, if proper means be taken to mark the motion of the spring (Fig. 1) in its course of extending itself, that, during this extension, 10 of the rings of which it is composed open to the left, in the direction of B's motion, and only one in the direction of A, while the point *a* (Fig. 2) has remained at rest. The spring has acted from a plane, passing through the point *a*, as from a base, to the right and to the left. Now will any one say, that, when 10 out of 11 parts of the spring have acted wholly in the direction of B, the effect of the spring has been equally distributed between A and B ?

Suppose we cut the spring in two at the point *a*, and fix the cut end of the larger part to an immovable plane, as in Fig. 3, and, placing the ball B against it when compressed, suffer it to extend itself against B. The velocity of motion that it will communicate to B will be precisely equal to that produced under the conditions of Figs. 1 and 2; and if we take the remaining coil of the spring, and fix it to the opposite side of the plane, as in Fig. 3, and suffer it to extend itself, it will produce the same motion in A that was produced in the first experiment. Now it must be evident that the 10 rings of the spring contain 10 times as much elastic force as the 1 ring, and that, as no motion whatever is communicated to the plane by their action, all the force of each

Fig. 3.



spring must be communicated to the bodies A and B, against which they act respectively; and as these forces are to each other as 1 to 10, so the forces of A and B must be as 1 to 10. But the velocities of A and B are to each other the same that they were in the first experiment. Hence it follows, that the forces produced in A and B by the unbending of the spring in the first experiment were not equal, but that B received 10 parts of the force and A 1 part; and this distribution corresponds exactly with the product of the mass into the square of its velocity.

For the mass of A = 10 and its velocity = 1; then $10 \times 1^2 = 10$ for the force of A; and the mass of B = 1 and its velocity = 10; then $1 \times 10^2 = 100$ for the force of B.

We shall find, moreover, that the forces of A and B, under the velocities produced by the action of the spring, are to each other inversely as the masses of A and B; that is, at the instant that they leave the spring,

Force of A : Force of B :: Mass of B : Mass of A.

Or, the force of B is as much greater than the force of A as the mass of A is greater than the mass of B. Again, if we neglect entirely the masses of A and B, or the quantity of the mass in each, we shall find that their forces are to each other as their velocities directly, and not as the squares of their velocities. Thus, we have found the force of B at the instant that it leaves the spring to be ten times as great as that of A. This was found by multiplying the mass of each into the square of its velocity. But the velocity of B is ten times as great as that of A. If therefore we neglect the masses, as factors, with both bodies, and take the force of each, not as the square of its velocity, but as its velocity simply, we have the same comparative results; that is,

Force of B : Force of A :: Velocity of B : Velocity of A.*

This last statement is not given as favoring the Newtonian measure of force. For the mass of the body is, in that, always taken as an element or factor of the force, and it can be struck out only in a case like the one here stated; where, on striking it out, the value of another factor of the force, namely the velocity, is evolved from the square to its simple power.

It cannot fail to be understood, that whenever two bodies, as A and B in the preceding example, receive the action of elasticity impelling them in opposite directions, the neutral point, as at *a*, Fig. 2, will be placed so as to divide the elastic force into two parts, having the same proportion to each other that the parts into which a line connecting the centres of gravity of the two bodies will be divided by their common centre of gravity. Thus in the example herein given, if A contained 100 times the mass of B, the spring would be divided by the point *a* so as to give 1 part to A and 100 parts to B; and if the mass of A be made infinite with regard to B, so as to be immovable, as in the plane, Fig. 3, the point *a* coincides with the end of the spring, and all the force is communicated to B.

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* Indeed, this amounts to no more than dividing the two sides of an equation by the same number.

A very striking example for illustrating the principles of this analysis, and displaying the force of elasticity in its highest form, may be imagined under the following order of preparation. Divide the charge of powder, intended for a piece of ordnance, by a diaphragm of iron, or any substance that will stand the heat, no matter how light and thin it be, into two parts, that shall be to each other as the weight of the gun is to that of the ball. Load the piece by placing the end of the cartridge containing the smaller portion of the charge at the bottom of the chamber, the ball being placed over the larger portion of the charge. Now if the fire be communicated to both parts of the charge at the same instant, the diaphragm must, however light, remain perfectly stationary in space, being subjected to equal pressures in opposite directions upon its two sides, while the gun and the ball are driven from it in opposite directions, with velocities inversely as their weights. Here, as the cannon is often 200 times as heavy as the ball, the quantity of powder before the diaphragm, and actually expended in producing the motion of the ball, is 200 times as great as that expended in producing the motion or recoil of the gun, though the pressure upon both is the same in intensity and in time; but the quantity of the force communicated to the ball must be, under these conditions, directly as the powder expended to produce it; and yet if we take the force of the ball and that of the gun in its recoil, respectively, as the product of the mass into its velocity simply, we shall make them equal, one to the other; that is, 1 oz. of powder acting from one side of a plane, equal to 200 oz. acting from its opposite side. But, by the measure of the square of the velocity, we have, mass of ball 1, velocity 200, and $1 \times 200^2 = 40{,}000$ for the force of the ball; mass of gun 200, velocity 1, and $200 \times 1^2 = 200$ for the force of the gun's recoil; which corresponds exactly with the expenditure of the powder. It may again be observed that these forces correspond, inversely, with the masses; for $200 : 40{,}000 :: 1 : 200$.

It can hardly be necessary to remark, that in any of the preceding instances we may reverse the direction of the motion, and consider the bodies as made to approach towards each other, rather than to recede from each other. Thus if A and B, when placed asunder as in Fig. 2, are drawn towards each other by a distended spring, or by any kind of attraction, or gravitation, inherent in the bodies themselves, the force excited in, or transferred to, each will be inversely as the masses of the bodies respectively, and this will be the case whether the elastic or attractive force be constant or variable. Hence, if the orbital motions of a satellite and its primary were destroyed, they would approach their common centre of gravity with forces *directly* as their velocities and *inversely* as their masses (*see* p. 367); and if another body were placed at their point of meeting, such body would receive a shock from the satellite as much greater than that received from the primary, as the velocity of the satellite was greater than that of the primary, or as the mass of the primary was greater than that of the satellite.

Hence in all those operations in the arts by which the form of unelastic bodies is changed by collision or percussion, as in the action of the hammer or axe, the efficiency of the instrument depends upon the inertia of the body against which it acts. Thus, if a block or anvil be suspended by a chain, or made to rest upon a spring, and a piece of lead or any soft and unelastic body be placed against or upon it and struck by a hammer, the force of the hammer will be divided between the lead and the bed or anvil; that is, in changing the form of the lead, and in giving motion to it and the anvil; and the distribution of the force, or the proportions which go to the different bodies involved in the action, may be found as follows: —

Let the mass of the hammer $= m$.
Its velocity before the blow $= v$.
Its velocity after the blow $= v'$.
The mass of the bodies struck $= m'$.
Velocity after the stroke $= v''$.

Then the *force* of the hammer before the stroke will be $m\,v^2$. Of this force there will remain in the motion of the masses after the stroke

$$m\,v'^2 + m'\,v''^2$$

and $m\,v^2 - m\,v'^2 - m'\,v''^2$ must be *absorbed* in changing the form of the lead.

If we neglect entirely the effect of elasticity, which we may do for most practical purposes, in both masses, then when m and m' are equal, one quarter of the force will remain in the motion of m, one quarter will be communicated to produce motion in m', and one half will go to change the form of the soft body. So, if the mass of m' be infinitely greater than that of m, or, not being greater, if m' be made to move in a direction opposite to the motion of m, and with a velocity such that m' and m remain at rest after the collision, then the whole force will be absorbed in changing the form of the soft body.

What is here stated may be applied to the whole subject of percussion or collision.